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Physics 39907 Computational Physics
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Problem Set 5

Question 1. 2×2 *Matrices:* Find explicit examples of 2×2 matrices with the following properties.

(1) $A^2 = -I$ for A with all real entries.
(2) $P^2 = I$ for $P \neq I$.
(3) $LU = UL$ for U upper- and L lower-triangular.
(4) $LU \neq UL$ for U upper- and L lower-triangular.
(5) $BC = -CB$ for $BC \neq 0$.
(6) $D^2 = 0$ with all elements of D are non-zero.

Question 2. *LDU factorization:* Find the values of w, x, y, z such that:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = LDU = \begin{bmatrix} 1 & 0 \\ w & 1 \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}$$

(1) Can all matrices A be decomposed this way? What are the conditions on a, b, c, d for this decomposition to work (or not work)?
(2) Assuming the decomposition exists, what are the conditions on w, x, y, z for A^{-1} to exist?
(3) Show that:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

(4) What is the condition on A for the inverse to exist? How does it compare with the conditions on w, x, y, z above?
(5) MATLAB has a function `[L,U]=lu(A)` that returns the LU decomposition. Make a new MATLAB function that returns the $A = LDU$ decomposition: `[L,D,U]=ldu(A)`. Here is a start:

```
function [L,D,U]=ldu(A)
% ldu <LDU decomposition of a matrix.>
% Usage:: [L,D,U]=ldu(A)
%

% revision history:
% 10/26/2023 Mark D. Shattuck <mds> ldu.m

%% Main
[L,U]=lu(A);
D=???
U=???
```

(6) Not all matrices A can be decomposed into $A = LDU$, where L is lower triangular. However, there is always a permutation of A which can be factored. So there is a always a permutation matrix P, a lower-triangular matrix L and an upper-triangular matrix U such that $PA = LU$. Further there is a unique factorization if the all of the diagonals of L are 1. Notice that $P^2 = I$ and so $A = PLU$.

So the MATLAB command `[L,U]=lu(A)` actually returns `L=PL`. Show that:

$$\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$$

does not have an LU decomposition and find its PLU decomposition and its $PLDU$ decomposition.

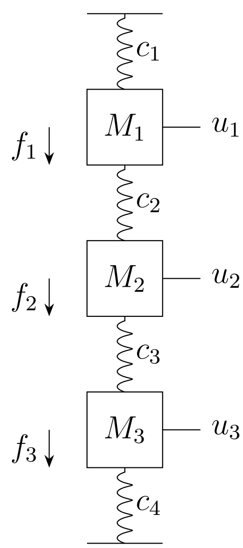
Question 3. Linear Equations: Find the linear combination of these 3 vectors:

$$u_1 = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}, u_2 = \begin{bmatrix} 2 \\ -4 \\ 9 \end{bmatrix}, u_3 = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$$

which gives

$$b = \begin{bmatrix} -2 \\ 2 \\ 9 \end{bmatrix}.$$

Question 4. Energy in a spring system: Consider a 3 mass 4 spring system with fixed-fixed boundaries separated by 1:



In class, we found the matrix representation of this system:

$$\begin{aligned} e &= \Delta u, \\ T &= Ce, \\ f &= \Delta^T T. \end{aligned}$$

(1) What are the sizes of each matrix e, Δ, u, T, C, f?
(2) Assuming the applied forces are constant, write the total work done on the system for a displacement of

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

directly in components and in matrix notation.

(3) Find the change in potential energy of the springs for the same displacement u in both component form and matrix form.
(4) Use the work-energy theorem to find the total potential energy of the system after a displacement u with forces f.
(5) Show that the minimum potential energy corresponds to the force balance solution we derived in class.
(6) Find and plot the displacements for the case that $c_m = 1 + m/2$, with masses $M_n = N - n$, where $N = 4$, `n=1:N-1`, `m=1:N` for the case shown and the forces are from gravity i.e., $f_n = M_n g$ and $g = 1$. You can plot the displacements against n.
(7) Find the solution for $N = 20$.