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Physics 35100 Mechanics
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Problem Set 2

Question 1. Two masses m_1 and m_2 are connected by a spring with spring constant K. The masses are confined to the the x-y plane and have position $\vec{x}_1(t)$ and $\vec{x}_2(t)$. The potential energy stored in the spring is:

$$V(l) = \frac{1}{2}K(l - l_0)^2,$$

where $l = \left[\vec{l} \cdot \vec{l}\right]^{\frac{1}{2}} = [(\vec{x}_2 - \vec{x}_1)^2]^{\frac{1}{2}}$ and $\vec{l} = l\hat{l} = \vec{x}_2 - \vec{x}_1$.

(1) Make a sketch of the system with m_1, m_1, $\vec{x}_1$, $\vec{x}_2$ and, $\vec{l}$ labeled.

(2) Given two vectors $\vec{a} = (a_x, a_y)$ and $\vec{x} = (x, y)$ such that $\vec{a}$ is a constant with respect to $\vec{x}$ (i.e., $\frac{\partial a_x}{\partial x} = 0$, $\frac{\partial a_x}{\partial y} = 0$, $\frac{\partial a_y}{\partial x} = 0$, and $\frac{\partial a_y}{\partial y} = 0$). Show that:

$$\frac{\partial(\vec{a} \cdot \vec{a})}{\partial \vec{x}} = \left(\frac{\partial(\vec{a} \cdot \vec{a})}{\partial x}, \frac{\partial(\vec{a} \cdot \vec{a})}{\partial y}\right) = (0, 0) = \vec{0}$$

(3) Given two vectors $\vec{a} = (a_x, a_y)$ and $\vec{x} = (x, y)$ such that $\vec{a}$ is a constant with respect to $\vec{x}$ (i.e., $\frac{\partial a_x}{\partial x} = 0$, $\frac{\partial a_x}{\partial y} = 0$, $\frac{\partial a_y}{\partial x} = 0$, and $\frac{\partial a_y}{\partial y} = 0$). Show that:

$$\frac{\partial(\vec{a} \cdot \vec{x})}{\partial \vec{x}} = \left(\frac{\partial(\vec{a} \cdot \vec{x})}{\partial x}, \frac{\partial(\vec{a} \cdot \vec{x})}{\partial y}\right) = (a_x, a_y) = \vec{a}$$

(4) Given the vector $\vec{x} = (x, y)$. Show that:

$$\frac{\partial(\vec{x} \cdot \vec{x})}{\partial \vec{x}} = \left(\frac{\partial(\vec{x} \cdot \vec{x})}{\partial x}, \frac{\partial(\vec{x} \cdot \vec{x})}{\partial y}\right) = (2x, 2y) = 2\vec{x}$$

(5) Use (2)-(4) and the chain rule to show that the vector function:

$$\frac{\partial l}{\partial \vec{x_1}} = \left(\frac{\partial l}{\partial x_1}, \frac{\partial l}{\partial y_1}\right) = -\hat{l}$$

(6) Use (2)-(5) and the potential to find the vector force on each particle. Write down 2 vector equations using Newton's second law for the two particle, and show that the two vector forces obey Newton's third law.

(7) Write the equation for the center of mass vector $\vec{X}_{cm}$ and use (6) to find the acceleration of the center of mass vector $\ddot{\vec{X}}_{cm}$.

(8) Consider the change of variables:

$$\vec{X}_{cm} = \frac{m_1\vec{x}_1 + m_2\vec{x}_2}{m_1 + m_2}$$
$$\vec{l} = \vec{x}_2 - \vec{x}_1$$

Find the inverse transformation: $\vec{x}_1 = \vec{x}_1(\vec{X}_{cm}, \vec{l})$ and $\vec{x}_2 = \vec{x}_2(\vec{X}_{cm}, \vec{l})$

(9) Write down the total energy E using the kinetic $T(\dot{\vec{x}}_1, \dot{\vec{x}}_2)$ and potential $V(\vec{x}_1, \vec{x}_2)$, then convert to the new variables from (8), $T(\dot{\vec{X}}_{cm}, \dot{\vec{l}})$ and $V(\vec{X}_{cm}, \vec{l})$. These definition may be useful: Total mass, $M = m_1 + m_2$ and reduced mass, $\mu = m_1 m_2/M$.

(10) Use the equation for the length vector $\vec{l} = \vec{x}_2 - \vec{x}_1$ and (6) to show that the acceleration of length vector is:

$$\ddot{\vec{l}} = -\frac{K}{\mu}(l - l_0)\hat{l}$$

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(11) Find $\ddot{\vec{l}} \cdot \vec{l}$ and $\ddot{\vec{l}} \times \vec{l}$ in complex polar coordinates $\vec{l} = le^{i\theta}$, where $i = \sqrt{-1}$. Where $\vec{u} \times \vec{v}$ is the 2D scalar cross product defined in terms of the ordinary 3D vector cross product $\vec{u} \times \vec{v} = [(u_x\hat{x} + u_y\hat{y} + 0\hat{z}) \times (v_x\hat{x} + v_y\hat{y} + 0\hat{z})] \cdot \hat{z}$. Some useful identities:

- For any vectors $\vec{u} = (u\cos\theta, u\sin\theta)$ and $\vec{v} = (v\cos\phi, v\sin\phi)$ represented as a complex numbers $U = ue^{i\theta}$ and $V = ve^{i\phi}$, then $U^*V = uve^{i(\phi-\theta)} = uv\cos(\phi - \theta) + iuv\sin(\phi - \theta) = \vec{u}\cdot\vec{v} + i\vec{u}\times\vec{v}$, where $U^* = ue^{-i\theta}$ is the complex conjugate of U.
- $ue^{i\theta} = u(\cos\theta + i\sin\theta)$.
- $\Re(ue^{i\theta}) = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = u\cos\theta$.
- $\Im(ue^{i\theta}) = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = u\sin\theta$.

For example, find $\dot{\vec{l}} \cdot \vec{l}$ and $\dot{\vec{l}} \times \vec{l}$:

$$\vec{l} = le^{i\theta}$$

$$\dot{\vec{l}} = \dot{l}e^{i\theta} + il\dot{\theta}e^{i\theta} = (\dot{l} + il\dot{\theta})e^{i\theta}$$

$$\dot{\vec{l}} \cdot \vec{l} + i\dot{\vec{l}} \times \vec{l} = (\dot{l} - il\dot{\theta})e^{-i\theta}le^{i\theta} = (\dot{l} - il\dot{\theta})l$$

$$\dot{\vec{l}} \cdot \vec{l} = l\dot{l}$$

$$\dot{\vec{l}} \times \vec{l} = -l^2\dot{\theta}$$

(12) Dot $\vec{l}$ with both sides of (10) and use (11) to give an equation of motion for $\ddot{l}$ in terms of l and θ.

(13) Cross $\vec{l}$ with both sides of (10) and use (11) to give a conserved quantity $L = \mu l^2\dot{\theta}$. (hint: Calculate $\dot{L}$.) L is the angular momentum of the system and $I = \mu l^2$ is the moment of inertia, which plays the role of mass in rotational problems. Using I, $L = I\dot{\theta}$.

(14) Show that $E = \frac{1}{2}\mu\dot{l}^2 + \frac{1}{2}\mu(l\dot{\theta})^2 + \frac{1}{2}K(l - l_0)^2$ is a conserved quantity. E is the total energy of the system.

(15) Use $L = \mu l^2\dot{\theta}$ to obtain a second-order non-linear differential equation of motion for l which depends only on l, $\dot{l}$, etc and constants (i.e., eliminate θ, $\dot{\theta}$, etc).

(16) The equation in (15) can not be solved analytically. To get an idea of the motion, use $L = \mu l^2\dot{\theta}$ to eliminate $\dot{\theta}$ in E and show that:

$$\dot{l} = \pm\left[\frac{2E}{\mu} - \left(\frac{L}{\mu}\right)^2 l^{-2} - \frac{K}{\mu}(l - l_0)^2\right]^{\frac{1}{2}}.$$

Show that the dimensionless equation using $[L] = l_0$, $[M] = \mu$, and $[T] = \sqrt{\mu/K}$ has the form:

$$\dot{\lambda} = \pm\left[\epsilon - \Lambda^2\lambda^{-2} - (\lambda - 1)^2\right]^{\frac{1}{2}}.$$

and give the values of the dimensionless energy ϵ and angular momentum Λ in terms of the energy E, angular momentum L, K, μ, and l_0. Plot both branches of $\dot{\lambda}$ vs. λ for a few values of ϵ and Λ and explain the dynamics. Ignore any values of $\dot{\lambda}$ that are imaginary.