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Physics 35100 Mechanics
September 28, 2022

Problem Set 3

Question 1. *Impulse-Momentum Theorem* The impulse $\vec{J}(t_1, t_2)$ over the time interval t_1 to t_2 of a force $\vec{F}(t)$ is defined as:

$$\vec{J}(t_1, t_2) = \int_{t_1}^{t_2} \vec{F}(t)\, dt.$$

(1) Using Newton's Law for the momentum $\vec{p}$ of a particle of mass m and the fundamental theorem of calculus show that $\Delta\vec{p} = \vec{p}(t_2) - \vec{p}(t_2) = \vec{J}(t_1, t_2)$. This is the Impulse-Momentum Theorem: The change in momentum is equal to the Impulse.

(2) Show that the average force over the interval from t to $t + \Delta t$, $\vec{F}_{avg}$, times the size of the interval Δt is equal to the impulse $\vec{J}(t, t + \Delta t) = \vec{F}_{avg}\Delta t$, and therefore:

$$\frac{\Delta\vec{p}}{\Delta t} = \vec{F}_{avg}.$$

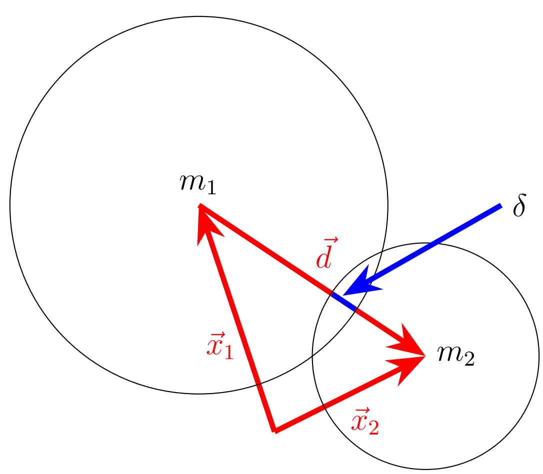
This discrete version of Newton's Law is not an approximation and is very useful when forces come in short bursts and otherwise there are no net forces.

Question 2. *Impulse of an elastic collision* Consider two disks with diameters D_1 and D_2 and masses m_1 and m_2 confined to the the x-y plane and have positions $\vec{x}_1(t)$ and $\vec{x}_2(t)$. When the disks are not in contact the potential between them is zero. When they are in contact the potential is determined by the overlap distance δ:

$$V(l) = \begin{cases} \frac{1}{2}K(l-D)^2 & l \le D \\ 0 & l > D \end{cases},$$

$$V(l) = \begin{cases} \frac{1}{2}K\delta^2 & \delta \ge 0 \\ 0 & \delta < 0 \end{cases},$$

where $l = \left[\vec{l} \cdot \vec{l}\right]^{\frac{1}{2}} = \left[(\vec{x}_2 - \vec{x}_1)^2\right]^{\frac{1}{2}}$, $\vec{l} = l\hat{l} = \vec{x}_2 - \vec{x}_1$, $D = (D_1 + D_2)/2$, and $\delta = D - l$ is the overlap distance.



(1) Defining a coordinate transformation to change to a frame of reference moving with particle 1 gives:

$$\vec{l} = \vec{x}_2 - \vec{x}_1,$$
$$\vec{X}_{cm} = \frac{m_1\vec{x}_1 + m_2\vec{x}_2}{M},$$

where $\vec{X}_{cm}$ is the center of mass, and $M = m_1 + m_2$ is the total mass. The inverse transformation is:

$$\vec{x}_1 = \vec{X}_{cm} - \frac{m_2}{M}\vec{l},$$
$$\vec{x}_2 = \vec{X}_{cm} + \frac{m_1}{M}\vec{l}.$$

The total kinetic energy is:

$$T = \frac{1}{2}M\dot{\vec{X}}_{cm}^2 + \frac{1}{2}\mu\dot{\vec{l}}^2,$$

where $\mu = \frac{m_1 m_2}{M}$. Find the Lagrangian of the system $L(\vec{X}_{cm}, \dot{\vec{X}}_{cm}, \vec{l}, \dot{\vec{l}})$ during a collision and when the particles are not overlapped.

(2) Find the generalized momentum $\vec{P}_{cm}$ conjugate to $\vec{X}_{cm}$. Does $\vec{P}_{cm}$ depend of whether the particles are in contact? Why or why not?

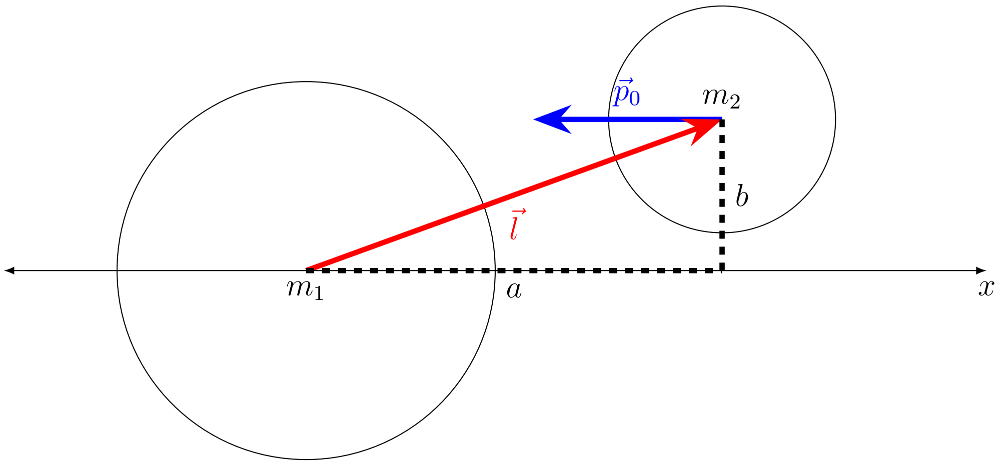
(3) Find the equation of motion for $\vec{X}_{cm}$ and show that $\vec{P}_{cm}$ is conserved. Does the result depend of whether the particles are in contact? Why or why not?

(4) Find the generalized momentum $\vec{p}$ conjugate to $\vec{l}$. Does $\vec{p}$ depend of whether the particles are in contact? Why or why not?

(5) Find the equation of motion for $\vec{l}$. When is $\vec{p}$ conserved? Does the result depend of whether the particles are in contact? Why or why not? (Hint compare with the spring system from problem set 2, or note: $\frac{\partial l}{\partial \vec{l}} = \frac{\partial}{\partial \vec{l}}(\vec{l}\cdot\vec{l})^{1/2} = 1/2(\vec{l}\cdot\vec{l})^{-1/2}2\vec{l} = \vec{l}/l = \hat{l}$.)

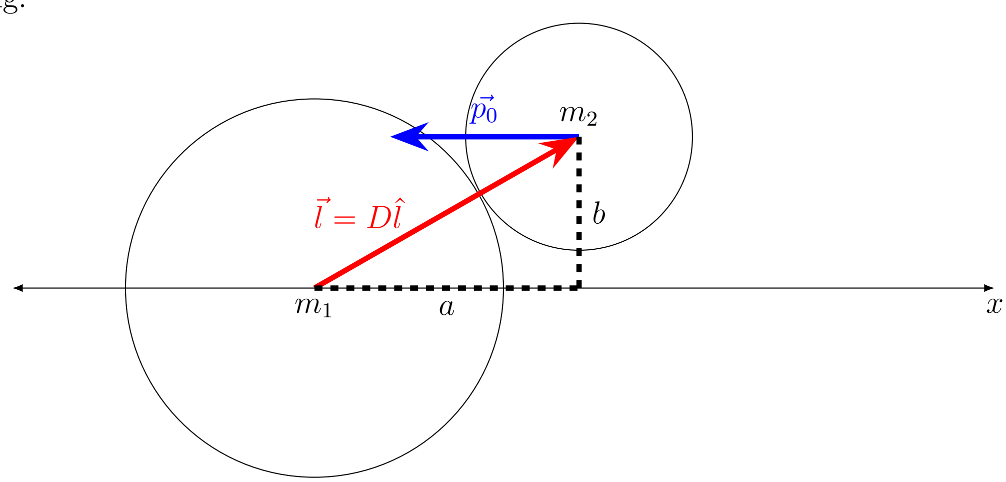
(6) Express the kinetic energy T in terms of the momenta $\vec{P}_{cm}$ and $\vec{p}$ and show that T is conserved wnen the particles are out of contact and that T_0 before a collision is equal to T_f after a collision using the fact that to the total energy E=T+V is always conserved.

(7) Before the two particles collide particle 1 is stationary at the origin in its frame of reference, and particle 2 is at position $\vec{l}$ moving with an initial constant momentum $\vec{p}_0$. Since the orientation of the system does not matter, we can simplify by rotating until particle 2 is on the right moving in the $-\hat{x}$-direction. Then we arrive at the following general situation before a collision:



The condition for a collision to occur is that $\vec{p}_0 \cdot \hat{x} > 0$ and $|b| < D$, where b is called the impact parameter. Explain what happens when these conditions are not met.

(8) As time progresses the particles move together, and at the point of collision, $\vec{l} = D\hat{l}$, and we have the following:



Set the collision time $t = 0$ and find the x and y components of $\vec{l}(t=0) = \vec{l}_0$, $l_0 = \left|\vec{l}_0\right|$ and $\hat{l}_0 = \vec{l}_0 / l_0$ in terms of b, and D .

(9) If we assume that the collision time Δ is very short and K is very large, then just after the collision the position of particles will still be the same, but the momentum will have changed due to the impulse from the collision. Use the Impulse-Momentum theorem to express the momentum after the collision $\vec{p}_f$ in terms of initial momentum $\vec{p}_0$ and the impulse $\vec{J}$.

(10) Using the definition of $\vec{J} = J\hat{J}$ determine the direction $\hat{J}$ of $\vec{J}$, assuming $\hat{l} = \hat{l}_0$ is constant during the collision.

(11) Use energy conservation $T(\vec{P}_{cm}, \vec{p}_f) = T(\vec{P}_{cm}, \vec{p}_0)$ to find J and $\vec{p}_f$ as a function of $\vec{p}_0$ and $\hat{l}_0$.

(12) Use the equation for $\vec{J}$ to add a geometric interpretation of $\vec{J}$ and $\vec{p}_f$ to the figure in (8). (Hint: A natural place to add $-\vec{J}/2$ is with its tail at the center of m_2. Add a line going through m_2 perpendicular to $\vec{l}$. Then add $\vec{J}$ to $\vec{p}_0$ to find $\vec{p}_f$.)

(13) To calculate the impulse $\vec{J}$ directly, we can focus on a simpler problem where $b = 0$. Solve the equations of motion for $\vec{l}$ when $b = 0$ and $\vec{l}(0) = D\hat{x}$, $\dot{\vec{l}} = -v_0\hat{x}$. Use the solution to directly calculate $\vec{J}$ by integrating the force over the collision time.

Question 3. *Double Springs* Mass m_1 at position $\vec{x}_1$ and m_2 at position $\vec{x}_2$ are connected by a spring with spring constant K_1 and rest length L_1. m_1 is also connected by a spring to a fixed point at position $\vec{x}_0 \equiv \vec{0}$ in an inertial frame by a spring with spring constant K_0 and rest length L_0. There are no external forces.

(1) Draw a labeled diagram of the system.

(2) Find the Lagrangian $L(\vec{l}_1, \dot{\vec{l}}_1, \vec{l}_2, \dot{\vec{l}}_2)$ where $\vec{l}_1 = \vec{x}_1 - \vec{x}_0$ and $\vec{l}_2 = \vec{x}_2 - \vec{x}_1$.

(3) Find the momenta conjugate to $\vec{l}_1$ and $\vec{l}_2$ and the equations of motion, and discuss how to solve them including what initial conditions would be needed.